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COMMENTARY PAPER

Neuroscientific studies of mathematical thinking and learning: a critical look from a mathematics education viewpoint

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Abstract In this commentary we take a critical look at the various studies being reported in this issue about the relationship between cognitive neuroscience and mathematics, from a mathematics education viewpoint. After a discussion of the individual contributions, which we have grouped into three categories—namely neuroscientific studies of (a) children's numerical magnitude representation, (b) arithmetical thinking, and (c) more advanced mathematical thinking—and which nicely document the scientific progression being made within the domain of educational neuroscience applied to the domain of mathematics education during the last 5 years, we point to some general caveats that should be considered in future research.

Keywords Neuroscience · Numerical magnitude representation · Arithmetic · Advanced mathematical thinking

1 Introduction

The various contributions to this special issue, which is a nice follow-up of a similar issue being published in the same journal about 5 years ago (Stern & Schneider 2010), nicely illustrate how neuroscience can be applied to education to enhance our understanding, to specify our predictions, and to improve our interventions within the domain of mathematical cognition (De Smedt & Grabner 2015). In this commentary we reflect upon the various studies being reported in this special issue, which we have grouped into three categories: neuroscientific studies of (a) children's numerical magnitude representation, (b) arithmetical thinking, and (c) more advanced mathematical thinking. While there is little doubt that these studies represent good examples of the scientific progression that has been made within the domain of educational neuroscience applied to the domain of mathematics education since the publication of the previous ZDM issue on this topic, some potential caveats should be considered, which are discussed briefly at the end of this commentary.

2 Neuroscientific studies of children's numerical magnitude representation

The dominant approach in cognitive neuroscience studies focusing on number development is based on the theory of the approximate number system (ANS), which relies on behavioral and neuroimaging measures of the distance effect in fast magnitude comparison as the main empirical source of evidence. There is however increasing criticism about the role of ANS in the development of number knowledge (Leibovich & Ansari 2016). Two of the articles of this special issue rely on the ANS approach but extend it beyond the basic comparison of symbolic (Arabic numerals) and non-symbolic magnitudes.

Merkley, Shimi, and Scerif (2016, this issue) use an artificial learning task as a tool to analyze how the learning of symbolic number knowledge may develop, focusing particularly on the development of the ordinal aspect of numbers. Undergraduate students were assigned to two treatment groups. Both groups were trained to connect artificial labels (non-words) and novel graphical figures. After that, one group was taught to connect each of the symbols

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to a magnitude and the other group was taught the order of the symbols without connection to the magnitudes. Based on the behavioral results of the magnitude comparison task there was a significant distance effect on reaction time in both groups: students were faster in far comparison. However only those students in the magnitude training group were more accurate in far comparison, while in the order training group there was no distance effect in accuracy. The ERP analysis found similar patterns of distance effects in both groups, which have previously been found with real numerical symbols. Most interestingly, in this study the emphasis is on numerical order. Studies on the development of number concept, particularly studies based on ANS-theory, have almost entirely focused on the cardinal aspect of the number system and the order aspect is seriously neglected. However, the design used in this study could have taken into account the successor function, the most important feature related to the order of natural numbers (Carey 2004). In addition, the artificial learning task and adult participants used in the study limit the ecological validity of the study and prevent any direct conclusions on how young children's number learning should be supported.

Pollack, Guerrero, and Star (2016, this issue) apply a priming method in studying the mental representation of literal symbols used in algebra. The study aims to explore the representation of literal symbols by analyzing if a priming distance effect and comparison distance effect, both widely known in comparisons of Arabic numerals, can be found in comparing literal symbols that the participants have learned to connect to certain numbers. As expected, both the priming distance effect and comparison distance effect were found among the participants when Arabic numerals were used. However, both distance effects disappeared when participants compared literal symbols after they had learned to connect them to certain numbers. In general, comparisons of literal symbols took more time than the comparison of Arabic numerals. Two different possible explanations are discussed in the article: (1) methods used in this study have limitations and are not able to catch the distance effects in literal symbols condition, or (2) literal symbols are related to networks of magnitude representations which do not result in distance effects. Particularly the later possible interpretation is interesting. It can be related to the use of literal symbols as such or to the situations where participants use more time to connect the literal symbol to their number knowledge before giving the answer. What is problematic in this study, however, is the questionable connection to algebra. Authors are aware that the way in which literal symbols are used in this study is representing very special cases of the use of literal symbols and does not deal with the variable nature of literal symbols fundamental for algebra. From a mathematics education viewpoint one of the problems in learning algebra is that students tend to think of literal symbols in terms of fixed natural numbers (Christou, Vosniadou & Vamvakoussi 2007).

The study of Schillinger, De Smedt, and Grabner (2016, this issue) also uses magnitude comparison tasks, but its aim is to deal with test pressure and anxiety in solving mathematical tests. The study is based on the use of the so called Stroop paradigm (fast comparison of Arabic numerals based on the congruent or incongruent magnitude or physical size of the symbols) and negative deflection in the EEG signal, which has proved to be more pronounced for erroneous (ERN) than for correct responses (CRN). These electrophysical indices are assumed to be related to the response monitoring. Participants of the study were university students. The suitable number of erroneous answers was controlled by regulating the instructions concerning the answering speed. Two conditions, low and high pressure, were used. In addition all participants filled out a test anxiety inventory. Behavioral data showed that performance pressure increased response time and surprisingly improved accuracy. The response times decreased with the level of test anxiety in the high pressure condition. The EEG data showed that both pressure condition and individual level of test-anxiety modulated the difference between CRN and ERN. In the discussion, the authors highlight the importance of simultaneous analysis of situational pressure and individual anxiety trait. This is a very interesting study and may contribute to the theoretical understanding of the role of test pressure and test anxiety on response monitoring. However, the very specific nature of the tasks used limits the educational relevance of the study. All participants knew the correct answers to all the tasks and errors were mainly mistakes due to the combination of time pressure and incongruent hints. Thus the monitoring of responses was very different from typical mathematics tests where the role of monitoring is to find out in an uncertain situation if the response is correct or not (Merenluoto & Lehtinen 2004).

Methodologically these three studies belong to the major trend in experimental psychology and neuroimaging studies on numerical cognition using reaction time designs. Fast responses and reaction time measures can tell something important about the representation of numbers, but as the dominating method they may also avoid seeing something important about the nature of number knowledge used in regular mathematical contexts. This resembles the early studies on memory using nonsense syllables to avoid the effect of prior knowledge and understanding. Findings of this type of research have been very robust, but the contribution to the theory of memory has been rather limited.

3 Neuroscientific studies on arithmetical thinking

While it seems that most neuroscientific studies deal with the topic of early numerical magnitude processing, researchers are also starting to address school-taught mathematical skills, particularly in the domain of whole and rational number arithmetic. Two articles address this topic.

Point of departure of Spüler, Walter, Rosenstiel, Gerjets et al. (2016, this issue) is that mathematics learners should be kept in an optimal range of cognitive workload to ensure that they are operating within their "zone of proximal development", and, that, therefore, it would be desirable to adapt the difficulty of (computer-based) training content to the learner's individual competencies. However, until now, adaptive computer-based learning environments primarily relied on learners' explicit interaction behavior for adaptation and not on direct evidence for learners' current cognitive state. The study of Spüler, Walter, et al. fits into a new line of research wherein attempts are made to use "brain-computer interfaces" that are expected to allow for a more direct and implicit monitoring of learners' states like cognitive workload by means of measuring specific neural correlates of these states. The researchers explored the possibility for such an interface of evaluating the signature of learners' oscillatory electroencephalogram (EEG), i.e., the specific patterns of electric signals produced by the brain, while solving blocked sets of arithmetic problems. To reach that goal, they followed a three-step procedure. First, they explored whether the task difficulty of arithmetic addition problems is reliably reflected in participants' individual EEG signatures. Second, they trained a regression model to predict item difficulty and thereby differentiate the presented arithmetic problems into three categories ranging from 'low' to 'medium' to 'high' difficulty. Third, they explored whether it is possible to reduce the number of electrodes needed for classification towards a more practical application.

While the study is characterized by a very challenging research objective, a carefully designed research methodology, and some promising results, from a mathematics educational perspective several queries can be raised. First, the study is based on the idea that the information content of an arithmetic task can be properly reflected by the Q-value (Thomas 1963), a measure that assumes that the essential parameters for problem difficulty in addition are the size of the addends involved and whether or not a carry is needed. Even though these are indeed two generally important task variables, it is striking that—as in most neuroscientific research of arithmetical thinking and learning—no attention is paid to the distinction between mental and written arithmetic. Whereas in written arithmetic, which is digit- and algorithm-based, these may indeed be the most important task features, in mental arithmetic, which is number- and heuristic-based (Verschaffel, Greer & De Corte 2007), other task features, such as how close the two given addends are to each other (e.g., 46 and 48) or how close they are to an easy number (e.g., to 50, 100 or 200) may be more important. Second, and related to the first problem, there is the response mode. Given that the participants were supposed to insert their answer on the computer, the question is whether they had to do that "from the left to the right" or "from the right to the left" (Verschaffel et al. 2007). This is not specified in the report, but this response mode may have great consequences on the nature of participants' solution strategies (written versus mental arithmetic) and, consequently, their accuracies. Third, it is notable that participants were adults, for (most of) whom adding two natural numbers is a well-learned skill. The question is what research can learn from such adult studies about how this skill develops in children and how that development can be optimally enhanced through (adaptive computer-based) instruction. Fourth, technical reasons did not only force the researchers to work with adults; they also forced them to apply a block design, in which task difficulty increased steadily from block to block. As acknowledged by the authors, this might have acted as a confounding factor, as effects such as fatigue may have influenced the EEG data, and, thus, their conclusion about prediction performance of their classification being based on problem difficulty. Finally, and more generally, mathematics educators may raise questions about the basic idea underlying the instructional approach followed by the researchers, namely designing a learning environment wherein the learner's learning trajectory is strictly individualized and controlled by an external entity. As for the first generation of intelligent tutoring systems dating from the 1990s, doubts can be raised about the value of such an instructional design wherein "the intelligence is put into the machine rather than in the learner" (De Corte, Verschaffel & Lowyck 1994). In the present case, this question seems even more pertinent as the system's adaptiveness is based on neuroscientific data of which the learner is completely unaware. Particularly from a broader self-regulatory learning perspective (De Corte, Mason et al. 2011), the educational implications of this approach need to be carefully considered. However, it should be clear that, notwithstanding these queries from a mathematics educational perspective, the search for neuroscientific measures of workload and other relevant measures that may affect mathematics learning, is of great theoretical importance and may lead to valuable education applications in the long term.

The second article, by Obersteiner and Tumpek (2016, this issue), deals with the use of eye-movement registration to unravel people's strategies for comparing fractions. During the past few years, several researchers have investigated whether fractions are processed in a holistic or componential manner, and how the type of strategy is affected by certain subject and task factors, by measuring participants' accuracies and response times on carefully designed sets of fraction comparison problems. However, these methods provide only an indirect measure of strategy use, and the alternative of verbal protocols risks being highly unreliable for this kind of task. Taking into account the available work on eye-tracking research in mathematical problems in general and in fractions in particular, the authors aimed to provide additional evidence for the claim that holistic and componential strategy use in fraction comparison depends on the type of fraction pair. Using eye-tracking in addition to response time measures allowed them to identify the fraction components that people took into account to make their choice. Of particular interest is that the researchers went beyond the mere computation of fixation durations on critical areas of interest, but also looked at saccades between specific fraction components reflecting the different strategies. Because the measuring technique required the researchers to investigate fraction comparison strategies under very demanding conditions in a highly controlled experimental setting, they decided to work with educated adults. Interestingly, their findings yielded further evidence that holistic comparison strategies play a larger role when fractions do not have common components compared to when they do have common components. According to the authors, their study is relevant to mathematics education, especially for diagnosing and helping younger learners with difficulties in solving fraction comparison problems who often struggle to verbalize their strategies, and for whom eye-tracking could help detecting strategy use in these students.

First, given the great accomplishments of this technique in other fields (e.g., reading and text comprehension), it is remarkable how little researchers in mathematics education have made use of eye tracking so far, particularly for the identification of strategies for operating on or comparing numbers. But, generally speaking, those studies were not completely successful, due to three major recurrent problems in the use of eye movements for strategy identification in the domain of mathematical cognition (see Verschaffel 2014). First, because the process of solving a mathematical problem typically not only consists of an execution phase, but also of an orientation and (possibly) a verification phase, which are experimentally hard to separate from each other. Second, because even if one were able to nicely isolate the execution phase, it frequently may not consist of the straightforward running of a single well-identifiable strategy. Third, because, once the presented information has been internally processed, the assumed close link between people's eye movements and their ongoing cognitive processing (which may hold for cognitive processes like reading) may get blurred. So, we agree with the authors that more research is needed to distinguish saccades that represent comparison strategies from saccades that represent reading or other processes. As a second comment, it is notable that participants were again adults who had to react to tasks under experimental conditions that were quite different from those of learners learning to compare fractions in a real school or out-of-school setting. Finally, while we agree that eye-tracking can be a very useful method to help detecting strategy use in learners in general and younger and/or mathematically weaker students (who often struggle to verbalize their strategies more than older and/or mathematically stronger learners) in particular, the question is how realistic and recommendable it is to think of a future wherein eye-tracking is part of a regular (remedial) mathematics classroom practice.

4 Neuroscientific studies on more advanced mathematical thinking

This special issue contains a few articles that go beyond elementary aspects of mathematics. The extensive attention that many cognitive and developmental psychologists spend to the rather basic aspects of numerical cognition (i.e. numerical magnitude and arithmetic) is typically justified by the fact that these are predictive for later mathematical achievement (e.g., De Smedt, Noël, Gilmore & Ansari 2013; Schneider, Grabner & Paetsch 2009; Siegler & Booth 2004). However, a recent meta-analysis (Schneider et al. 2016) shows that children's non-symbolic numerical magnitude processing abilities predict only 6 % of the individual differences in general math performance, and interventions on the early enhancement of children's basic numerical abilities often result in only small retention and transfer effects (Torbeyns, Obersteiner & Verschaffel 2012). Many mathematics educators (English & Mulligan 2013; Müller et al. 2012) argue that while mathematics as a discipline indeed uses numbers and arithmetic, they are not the core of mathematics. Mathematics is a science of patterns and relationships, and numbers and arithmetic are often even *avoided* when mathematical thinking occurs at a higher level: Letters are used that stand for a general number, and arithmetical operations cannot and do not longer need to be conducted. So, we strongly welcome the attention by neuroscientific researchers for more advanced mathematical thinking.

Of course, there are many possible subdomains to focus on when one moves in the direction of more advanced mathematical concepts. In this special issue, Waisman, Leikin, and Leikin (2016, this issue) focus on the properties of quadrilaterals, Leikin Waisman, and Leikin (2016, this issue) use so-called "insight problems" and part/whole area problems, Babai, Nattiv, and Stavy (2016, this issue) focus on the perimeter of polygons, and Vogel, Keller, Koschutnig, Reishofer, et al. (2016, this issue) on risk evaluation. In all tasks except the one used by Vogel et al., number and arithmetic play no—or at best a very minimal—role, which clearly distinguishes them from the other studies in this special issue.

In the risk evaluation task used by Vogel et al. (2016, this issue), however, the distinction is less straightforward: Participants had to indicate a position on a number line that corresponds to how concerned they are of getting cancer based on data provided in a visual array of white and black dots. This task comes extremely close to a number line estimation task as used in typical numerical magnitude understanding research. So, why would one consider this task as measuring a more advanced mathematical concept? The difference lies precisely in the framing and instruction of the task to indicate the experienced concern. The authors have used a contrasting task with exactly the same stimuli but with the framing and instructions typical for numerical magnitude estimation tasks. It is by making this clever contrast that the more advanced processes of risk perception and evaluation (the cognitive estimation of a probability, the emotional appreciation of the undesired outcome, and the weighing of both) can be determined. In the risk evaluation task by Vogel et al. (this issue), in principle there is no "correct" answer. High-numeracy participants in their study have a very accurate number line estimation but still systematically indicate a higher concern for getting cancer than low-numeracy students. This overestimation is not necessarily to be considered "incorrect" as this is a matter of subjective appreciation. In the other studies under consideration, one can objectively distinguish correct from incorrect answers. Therefore, for these studies it seems worth taking a further look at the tasks that were employed and at the possible reasons why participants would commit errors to these tasks:

Waisman et al. (2016, this issue) gave a task about logical inferences about the properties of quadrilaterals to adolescents. It is striking that no specific account is given on the knowledge that is supposed to be present in the participants, nor is there an explicit indication of the processes that the authors assume to take place when they commit an error. One can wonder what insights can be gained from observing differences in performance (and underlying neurological processes) by gifted or non-gifted and mathematically excelling or non-excelling students if from the start on we have no specific understanding of the actual underlying problem solving processes. We can assume that all participants had the required content-specific knowledge about properties of quadrilaterals. If this assumption is true, errors are committed in applying the rules of logic. However, these in their turn would need an explanation: Do participants insufficiently master some logic rules? Do they systematically misapply some rules? Or do they commit occasional, and thus "random", mistakes due to a lack of attention in a long series of trials?

A similar comment can be made to the Leikin et al. study (2016, this issue): The central task relates to "insight based problems", which they collected from a number of papers. Besides indicating that these were short problems whose solution strategies seemed to belong to a topic not obviously related to the problem and thus eliciting an "Aha!" moment, we get no further specification of the tasks, nor of the processes that are assumed to take place while solving. We have no indication about the errors that one can expect to be committed and the reasons for them. Still, basing ourselves on the example provided in the article ("A man bought a horse for $60 and sold it later for $70. Then he bought it back for $80 and sold it for $90. The money the man earned is …"), one can reasonably assume that the adolescent participants had all required knowledge to respond correctly. So the question remains, what errors would be committed by learners, and why?

This is somewhat different in the Babai et al. study (2016, this issue). Babai et al. depart from an explicit and general account of errors committed in mathematical and scientific reasoning (the inclination to apply an intuitive "More A–More B" rule) which is then specifically applied to the tendency to focus on the area of a rectangle (which is turned into a polygon) in order to make a judgment about the perimeter. Once more, one can assume that the adults in the study (the actual neuroscientific study is not reported in the article, but the intervention is claimed to be based on it) have all required knowledge to make appropriate judgments, but for some reason they sometimes make errors. In this specific case, the reason for errors is made explicit, namely the tendency in humans to rely on intuitive rather than analytic reasoning based on superficial, salient problem characteristics (see also Gillard et al. 2009; Leron & Hazzan 2006, 2009).

So, it is remarkable that both for the Waisman et al., Leikin et al., and Babai et al. studies, the tasks that were used had little ecological validity in the sense that the mathematical tasks provided to participants are not tasks of a particularly challenging nature, as they would for instance appear in school achievement tests. The tasks seem more suited to much younger learners. And with the exception of Babai et al., no account is provided as to which thinking processes would underlie correct and incorrect answers. One can even wonder whether the thinking processes that occurred in these neuroscientific studies are comparable to those in younger learners who are in the process of acquiring the involved mathematical concepts. Fischbein (1987), for instance, has shown that reasoning processes can qualitatively differ across ages, and some intuitions increase and others decrease with age.

A question that is often posed is what can be actually gained by collecting neuro-data on mathematical problem solving processes. An often made claim is that this type of data can corroborate existing theories and insights. This is certainly the case for the studies in this special issue. Waisman et al. and Leikin et al. for instance, did this for the insight that general intelligence as well as other factors (high effort, motivation) can explain high mathematical performance, and that they are interrelated but not identical to each other. Another claim that is made is that neuroscientific insights can be used to improve instruction. However, concrete indications for this claim are often missing. Babai et al. argue that their neuroscientific study led to a deeper understanding of students' reasoning processes, and consequently to improved teaching. However, it seems that the behavioral studies that they conducted besides the neuroscientific study were already sufficient to design the intervention study that was presented. Also, the conclusion that altering the mode or order of representations provides a tool for teachers to help overcome difficulties in students is not a novel one in the mathematics education community (e.g., Acevedo Nistal, Van Dooren, Clarebout, Elen & Verschaffel 2009; Elia, Panaoura, Eracleous & Gagatsis 2007), nor is it based on findings directly stemming from neuroscientific techniques. Leikin et al. argue that their findings that differentiate the processing of learning-based and insight-based tasks and the role of general intelligence versus mathematical excellence must be taken into consideration by math educators, when considering ability grouping and using insight-based tasks in the curriculum. However, they do not go further than this general recommendation, and we do not see how the reported study could provide concrete implications.

A pathway for future neuroscientific studies seems to lie in a focus on tasks for which there already is a detailed understanding and theoretical account of the problem solving processes that are expected to take place in the age group under investigation, which are backed up by empirical data, and in being modest in the implications that initial neuroscientific findings can have, besides refining or refuting this understanding of the underlying processes.

5 Conclusion

In 2010, an issue about the relationship between cognitive neuroscience and mathematics learning was published in ZDM (Stern & Schneider 2010). While there is little doubt that the studies presented in this new issue representing the state-of-the-art in this area of research anno 2016 represent good examples of the scientific progression being made within the domain of educational neuroscience applied to the domain of mathematics education, some potential caveats should be considered, which are discussed briefly at the end of this commentary. Actually, these caveats largely echo the major concerns raised by De Smedt and Verschaffel (2010) in their discussion of the papers published in the previous ZDM issue on the topic and by De Smedt and Grabner (2016) at the end of their excellent recent review of the state-of-affairs with respect to the applications of neuroscience to mathematics education in the *Oxford Handbook of Numerical Cognition*.

A first major challenge—repeatedly raised in the above commentary—deals with the tensions between the practical and technical constraints of the available neuroscientific methods and the need to achieve a sufficiently high ecological validity. These tensions concern both the participants tested in cognitive neuroscience studies (most of the existing body of data is still based on studies using adult populations) and the mathematical tasks used in these studies (most tasks are very elementary and differ substantially from the tasks typically solved in the classroom).

Second, according to De Smedt and Grabner (2015), most neuroscientific studies on the domain of mathematics education have investigated mathematical performance in relative isolation from the educational context. Also in most studies being reported in this special issue, participants' learning histories as well as their educational environment have been typically considered as nuisance variables that are ignored or controlled for. These variables are, however, crucial, as variability in these might have massive effects on brain structure and activity (De Smedt & Grabner 2015). Indeed, the development of mathematics cannot be studied in isolation from the learning context in which it is taught, as is documented by several decades of educational research. Without knowledge on how mathematics is learned and taught at school, cognitive neuroscientists are at the risk of running naïve experiments with little or no relevance to educational practice (De Smedt & Verschaffel 2010).

In sum, while the contributions to this special issue reflect the continuing growth and rapprochement of the disciplinary fields of cognitive neuroscience and mathematics education since the first ZDM issue on this topic, as well as the extension and sophistication of the repertoire of neuroscientific techniques being used and the diversification and complication of the kind of mathematical topics being addressed in the studies, it seems clear that several of the initial concerns of merging neuroscientific and educational research remain.

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